Matroids and their friends over $\mathbb{F}_1^{\pm}$-algebras

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Building Bridges Between $\mathbb{F}_1$-Geometry, Combinatorics and Representation Theory
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Matroids

Gian-Carlo Rota: Like many other great ideas, matroid theory was invented by one of the great American pioneers, Hassler Whitney. His paper flagrantly reveals the unique peculiarity of this field, namely, the exceptional variety of cryptomorphic definitions for a matroid, embarrassingly unrelated to each other and exhibiting wholly different mathematical pedigrees. It is as if one were to condense all trends of present day mathematics onto a single finite structure, a feat that anyone would *a priori* deem impossible, were it not for the mere fact that matroids do exist.

Matroids: the basis axiom

Let $E = [n] := \{1, \ldots, n\}$ be a finite set.

Definition

A matroid is a pair $M = (E, \mathcal{B})$, where the nonempty $\mathcal{B} \subseteq 2^E$ satisfies the following basis exchange axiom: if $B_1, B_2 \in \mathcal{B}$, then for every $x \in B_1 \setminus B_2$, there exists $y \in B_2 \setminus B_1$ such that $B_1 \setminus \{x\} \cup \{y\} \in \mathcal{B}$.

Bases are matroid-theoretic generalization of maximal independent sets. All bases have the same cardinality, called the rank of the matroid.

The basis exchange axiom is equivalent to a even stronger symmetric basis exchange axiom: we can make $B_1 \setminus \{x\} \cup \{y\}, B_2 \setminus \{y\} \cup \{x\} \in \mathcal{B}$ at the same time.

If $M = (E, \mathcal{B})$ is a matroid of rank r, then $M^* = (E, E \setminus \mathcal{B})$ is a matroid of rank $n - r$, called the dual matroid.

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One may also define matroids via circuit axioms.
Circuits are matroid-theoretic generalization of minimal dependent sets.

Definition

A matroid is a pair $M = (E, \mathcal{C})$, where the set of circuits $\mathcal{C} \subseteq 2^E$ satisfies the following axioms:
(a) Every $C \in \mathcal{C}$ is nonempty.
(b) Anti-chain. If $C_1 \subseteq C_2$ are in $\mathcal{C}$, then $C_1 = C_2$.
(c) Circuit elimination. If C_1 and C_2 are distinct circuits with $e \in C_1 \cap C_2$, then $(C_1 \cup C_2) \setminus \{e\}$ contains a circuit.

The circuits of the dual matroid M^* are called the cocircuits of M.

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Basis polytope of a matroid

Given a basis B of M, the indicator vector of B is $\vec{e}_B = \sum_{i \in B} \vec{e}_i \in \mathbb{R}^n$.
The basis polytope P_M of M is the convex hull of $\{\vec{e}_B | B \text{ is a basis of } M\}$.

Theorem [Gelfand-Goresky-MacPherson-Serganova, 1987]

A polytope P with vertices in $\{0,1\}^n$ is the basis polytope of a matroid if and only if every edge of P is parallel to $\vec{e}_i - \vec{e}_j$ for distinct i and j.

The vectors $\{\vec{e}_i - \vec{e}_j\}$ with distinct i and j form the root systems of type A, and the corresponding Coxeter groups are just the symmetric groups.
What are the corresponding matroids for other types of root systems and Coxeter groups?
We are interested in the type D case, and call them the orthogonal matroids.

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Orthogonal matroids

The ground set is still $E = [n]$. The symmetric difference of two sets is denoted $A \triangle B = (A \setminus B) \cup (B \setminus A)$.

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Definition (orthogonal matroid via basis polytope)

An orthogonal matroid on E is a polytope whose vertices are in $\{0, 1\}^n$ and whose edges are parallel to $\vec{e}_i \pm \vec{e}_j$ with distinct $i, j \in [n]$.

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Example

Matroids on $[n]$

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Represent orthogonal matroids by matrices

Consider the following skew-symmetric matrix over $\mathbb{Q}$:

$$A = \begin{pmatrix} 0 & -1 & -2 & 1 & 0 \\ 1 & 0 & 1 & 3 & 0 \\ 2 & -1 & 0 & 2 & -1 \\ -1 & -3 & -2 & 0 & 1 \\ 0 & 0 & 1 & -1 & 0 \end{pmatrix}.$$

The collection of all subsets $J \subseteq [5]$ where the $J \times J$ submatrix A_J is nonsingular is the set of bases of an orthogonal matroid.

In this example, we have $E = [5]$ and $\mathcal{B} = \{\emptyset\} \cup \binom{[5]}{4} \cup \binom{[5]}{2} - \{\{15\}, \{25\}\}$. Take $\emptyset$ and $\{1234\}$ in $\mathcal{B}$, and 1 in the symmetric difference $\{1234\}$. Then we can choose $2 \neq 1$ such that $\emptyset \triangle \{12\}, \{1234\} \triangle \{12\} \in \mathcal{B}$.

The row space of $(I_5|A)$ is an example of a maximal isotropic subspace of $\mathbb{Q}^{10}$.

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Orthogonal Grassmannians

Let K be a field. A subspace W of $V = K^{2n}$ endowed with a symmetric, non-degenerate bilinear form is called isotropic if $\langle W, W \rangle = 0$.

We will only consider the maximal isotropic subspaces, which have dimension n.

Maximal isotropic subspaces of V are parameterized by the orthogonal Grassmannian $OG(n, 2n) \subset \mathbb{P}^{2^n-1}(K)$. Its coordinates correspond to subsets of $[n]$, called the Wick coordinates.

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Proposition

$OG(n, 2n)$ is determined by some quadratic relations called the Wick relations, i.e., for all $J_1, J_2 \subseteq [n]$, if

$$J_1 \triangle J_2 = \{x_1 < x_2 < \cdots < x_m\},$$

then,

$$\sum_{k=1}^{m} (-1)^k \cdot X_{J_1 \triangle \{x_k\}} \cdot X_{J_2 \triangle \{x_k\}} = 0.$$

The simplest (nontrivial) ones are the 4-term Wick relations:

$$X_{Jabcd}X_J - X_{Jab}X_{Jcd} + X_{Jac}X_{Jbd} - X_{Jad}X_{Jbc} = 0,$$
$$X_{Jabc}X_{Jd} - X_{Jabd}X_{Jc} + X_{Jacd}X_{Jb} - X_{Jbcd}X_{Ja} = 0.$$

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Proposition

Given a maximal isotropic subspace $W \subseteq V$, the support of the associated Wick vector $w \in OG(n, 2n)$ is the set of bases of an orthogonal matroid.

An orthogonal matroid arising in this way is representable over K.

Theorem [Baker-Jin], using a theorem of [Nelson, 2018]

Asymptotically, 100% of orthogonal matroids are not representable over any field.

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A pointed monoid B is a set B together with an associative and commutative multiplication $\cdot : B \times B \to B$, and two elements $0, 1 \in B$ such that $0 \cdot a = 0$ and $1 \cdot a = a$ for all $a \in B$. We write ab for $a \cdot b$.

Let B be a pointed monoid. Identifying $0 \in B$ with the additive neutral element in $\mathbb{N}[B]$ defines a semiring $B^+ = \mathbb{N}[B]/\langle 0 \rangle$, which we call the ambient semiring of B. An ideal of B^+ is a subset I such that $0 \in I, I + I = I$, and $B \cdot I = I$.

Definition

A band is a pointed monoid B together with an ideal $N_B \subseteq B^+$ such that for every $a \in B$, there exists a unique element $b \in B$ such that $a + b \in N_B$.

We think of N_B as linear combinations of elements of B which 'sum to zero', and call it the null set of the band B. In this sense, we write $-a$ for the unique element $b \in B$ with $a + b \in N_B$, and call it the additive inverse of a.

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A band morphism is a multiplicative map $f : B \to C$ with $f(0) = 0$ and $f(1) = 1$ such that $\sum a_i \in N_B$ implies $\sum f(a_i) \in N_C$.

The regular partial field is $\mathbb{F}_1^{\pm} = \{0, 1, -1\}$, and the null set is $\{0, 1 + (-1), 1 + 1 + (-1) + (-1), \dots\}$. This is the initial object in the category of bands.

The final object is the trivial band $\{0\}$ in which $0 = 1$.

A ring R is naturally a band with null set $N_R = \{\sum a_i | \sum a_i = 0 \in R\}$.

An idyll is a band B with $0 \neq 1$ and $B^\times = B \setminus \{0\}$.

The final object for idylls is the Krasner hyperfield $\mathbb{K} = \{0, 1\}$ whose null set is $\{0, 1 + 1, 1 + 1 + 1, \dots\}$.

The category of bands has all limits, colimits, free algebras, and quotients. These will be useful studying representations of matroids and orthogonal matroids.

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The regular partial field is $\mathbb{F}_1^{\pm} = \{0, 1, -1\}$, and the null set is $\{0, 1 + (-1), 1 + 1 + (-1) + (-1), \dots\}$. This is the initial object in the category of bands.
The final object is the trivial band $\{0\}$ in which $0 = 1$.
A ring R is naturally a band with null set $N_R = \{\sum a_i | \sum a_i = 0 \in R\}$.
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The final object for idylls is the Krasner hyperfield $\mathbb{K} = \{0, 1\}$ whose null set is $\{0, 1 + 1, 1 + 1 + 1, \dots\}$.

The category of bands has all limits, colimits, free algebras, and quotients. These will be useful studying representations of matroids and orthogonal matroids.

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The prime k-ideals

Let B be a band.
A prime k-ideal of B is a subset $\mathfrak{p} \subseteq B$ such that $0 \in \mathfrak{p} \neq B, \mathfrak{p}B = \mathfrak{p}$, $B \setminus \mathfrak{p}$ is multiplicatively closed, and if $b + \sum b_j \in N_B$ with all $b_j \in \mathfrak{p}$, then $b \in \mathfrak{p}$.
The prime k-spectrum $\mathrm{Spec}^k(B)$ is defined to be the topological space whose points are prime k-ideals, with topology defined by $U_f = \{\mathfrak{p} : f \notin \mathfrak{p}\}$.

One can define localizations by any multiplicatively closed subset S of B.

Theorem

There is a presheaf $\mathcal{O}_X^k$ on $X = \mathrm{Spec}^k(B)$ such that $\mathcal{O}_{X,\mathfrak{p}} = B_{\mathfrak{p}}$.

Problem: this is in general NOT a sheaf: the local sections do not necessarily patch together to give a global section.

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A band space is a topological space X together with a sheaf of bands.
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Tracts are firstly introduced in [Baker-Bowler, 2019]. They are idylls (F, N_F) for which the null set is not required to be closed under addition.

Explicitly, a tract $F = (F, N_F)$ is an abelian group $F^\times$ (written multiplicatively) with $0 \neq 1$, together with an additive relation structure $N_F \subseteq \mathbb{N}[F]$ such that:

(T1) $0 \in N_F$, $1 \notin N_F$.

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Let F be a tract.
A Wick function on $[n]$ with coefficients in F is $\varphi : 2^{[n]} \to F$ such that:

(W1) φ is not identically zero.

(W2) All Wick relations are satisfied, i.e. for all $J_1, J_2 \in [n]$, we have

$$\sum_{k=1}^{m} (-1)^k \cdot \varphi(J_1 \triangle \{x_k\}) \varphi(J_2 \triangle \{x_k\}) \in N_F,$$

where $J_1 \triangle J_2 = \{x_1 < \cdots < x_m\}$.

Two Wick functions φ and ψ with coefficients in F are equivalent if $\varphi = c \cdot \psi$ for some nonzero $c \in F$.

We call an equivalence class of Wick functions an orthogonal matroid over the tract F, or simply an orthogonal F-matroid.

Proposition [Jin-Kim]

The support $\mathrm{Supp}(\varphi) := \{J \subseteq [n] : \varphi(J) \neq 0\}$ of a Wick function $\varphi : 2^{[n]} \to F$ is the set of bases of an orthogonal matroid.

We call it the underlying orthogonal matroid of φ, denoted $\underline{M}_{\varphi}$.

Orthogonal matroids over tracts

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Pushforwards

Proposition [Jin-Kim]

Let $f : F_1 \to F_2$ be a tract morphism. If φ is a Wick function with coefficients in F_1, then the composition $f \circ \varphi$ is a Wick function with coefficients in F_2. Moreover, $\underline{M}_\varphi = \underline{M}_{f \circ \varphi}$.

This gives us a pushforward operator f_* mapping orthogonal F_1-matroids to orthogonal F_2-matroids.

Let M be an orthogonal F-matroid. If we take $g : F \to \mathbb{K}$, the final object, then $g_*(M)$ is the same as the underlying orthogonal matroid of M.

We say an orthogonal matroid $\underline{M}$ is representable over a tract F if there exists an orthogonal F-matroid M' such that $g_*(M') = \underline{M}$.

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Products

Proposition [Jin-Kim]

Let F_1, F_2 be tracts, and let φ_1, φ_2 be Wick functions with coefficients in F_1, F_2, respectively, with the same underlying orthogonal matroid $\underline{M}$. Then $\varphi_1 \times \varphi_2 : 2^{[n]} \to F_1 \times F_2$ defined as $(\varphi_1 \times \varphi_2)(T) := (\varphi_1(T), \varphi_2(T))$ is a Wick function with coefficients in the product $F_1 \times F_2$ with underlying orthogonal matroid $\underline{M}$.

Regular orthogonal matroid

Theorem [Baker-Jin]

Let P be a partial field. Then $\varphi : 2^{[n]} \to P$ is a Wick function if and only if the support of φ gives an orthogonal matroid M and φ satisfies the 4-term Wick relations.

In the latter case, we say that φ is a weak representation of M over P.

Theorem [Geelen, 1996] [Jin-Kim]

Let M be an orthogonal matroid. Then the following are equivalent:

(i) M is representable over $\mathbb{F}_2$ and $\mathbb{F}_3$.

(ii) M is representable over the regular partial field $\mathbb{F}_1^{\pm}$.

(iii) M is representable over all fields.

Proof for the regular orthogonal matroid characterizations

(ii) $\Rightarrow$ (iii) is given by the unique tract morphism $\mathbb{F}_1^{\pm}$ to the field K.

(iii) $\Rightarrow$ (i) is trivial.

(i) $\Rightarrow$ (ii). If M is representable over $\mathbb{F}_2$ and $\mathbb{F}_3$, then it is representable over $\mathbb{F}_2 \times \mathbb{F}_3$. Take the (unique) group homomorphism $(\mathbb{F}_2 \times \mathbb{F}_3)^\times \to (\mathbb{F}_1^{\pm})^\times$, it gives a weak orthogonal $\mathbb{F}_1^{\pm}$-matroid whose underlying orthogonal matroid is also M. Since $\mathbb{F}_1^{\pm}$ is a partial field, the weak orthogonal $\mathbb{F}_1^{\pm}$-matroid is automatically a (strong) orthogonal $\mathbb{F}_1^{\pm}$-matroid. $\square$

We also give two new characterizations of regular orthogonal matroids without a specific minor M_4.

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Cryptomorphisms

We also have notions of orthogonal F-signatures and F-circuit sets that capture the circuit axioms of orthogonal matroids, and a notion of orthogonal F-vector sets that generalizes vectors of matroids over tracts [Anderson, 2019].

The cryptomorphism proof involves a homotopy theorem [Wenzel, 1995] on the 1-skeleton of the basis polytope, called the basis graph of the orthogonal matroid. Wenzel's result generalizes Maurer's homotopy theorem for matroids [Maurer, 1973].

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Homotopy theorem on the basis graph

The basis graph of an orthogonal matroid M has vertex set $\mathcal{B}$. Two vertices $B_1, B_2 \in \mathcal{B}$ are adjacent if $|B_1 \triangle B_2| = 2$, i.e. $B_2 = B_1 \triangle \{x, y\}$ for $x \neq y$.

For example, the basis graph of the orthogonal matroid associated to

$$A = \begin{pmatrix} 0 & -1 & -2 & 1 & 0 \\ 1 & 0 & 1 & 3 & 0 \\ 2 & -1 & 0 & 2 & -1 \\ -1 & -3 & -2 & 0 & 1 \\ 0 & 0 & 1 & -1 & 0 \end{pmatrix}$$

contains the following cycle.

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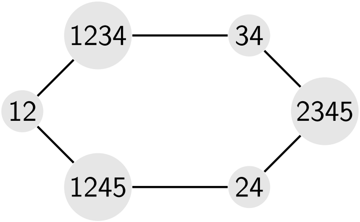

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Every cycle in the basis graph of an orthogonal matroid can be decomposed into cycles of length at most 4.

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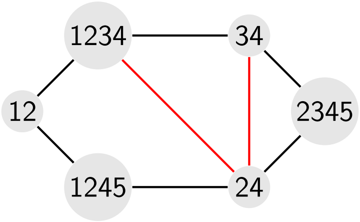

Back to matroids

When the support of a weak Wick function φ is a matroid, φ is also known as a weak Grassman-Plücker function.
Two weak Grassman-Plücker functions φ and φ' are in the same rescaling equivalence class if they are in the same orbit of the action of $T = (K^\times)^N$ on $\mathbb{G}(r, n)$.
We denote by $\chi_M^R(F)$ the set of rescaling equivalence classes of weak Grassman-Plücker functions over a pasture (i.e. a 3-term tract) F with support M.

Theorem [Baker-Lorscheid, 2021]

The functor χ_M^R is representable by a pasture F_M canocinally attached to the matroid M, i.e. we have $\chi_M^R(F) \cong \mathrm{Hom}(F_M, F)$.

We call F_M the foundation of M.

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Reformulation of Lafforgue's theorem

A matroid is called rigid if its basis polytope has no non-trivial regular subdivisions into other matroid polytopes.

Alex Fink's PhD thesis: Matroid subdivisions have made prominent appearances in algebraic geometry. [...] Lafforgue's work implies, for instance, that a matroid whose polytope has no subdivisions is representable in at most finitely many ways, up to the actions of the obvious groups.

Folklore Theorem, and [Baker-Lorscheid, 2023+]

If M is a rigid matroid, then $\chi_M^R(K)$ is finite for every field K.

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Now for orthogonal matroids...

Question

What can we say about the (rescaling) representation space $\chi_M^R(F)$ of an orthogonal matroid M?

Answer: χ_M^R taking a 4-term tract P to the set of rescaling equivalence classes of moderately weak orthogonal P-matroids with support M is representable by a 4-term tract, which by abuse of notation we denote again by F_M and call it the foundation of the orthogonal matroid M.

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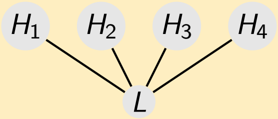
Presentation of F_M when M is a matroid

Theorem [Baker-Lorscheid, 2023+]

(i). The foundation of a matroid M is generated by configurations



where H_i's are distinct hyperplanes of M and L is a corank 2 flat contained in all H_i's.
We call such a configuration a universal cross ratio.

(ii). The relations between the universal cross ratios are inherited from embedded minors of M on at most 7 elements.

The proof combines Tutte's homotopy theorem and ideas from [Gelfand-Rybnikov-Stone, 1995].

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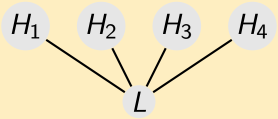

where H_i's are distinct hyperplanes of M and L is a corank 2 flat contained in all H_i's.
We call such a configuration a universal cross ratio.

(ii). The relations between the universal cross ratios are inherited from embedded minors of M on at most 7 elements.

The proof combines Tutte's homotopy theorem and ideas from [Gelfand-Rybnikov-Stone, 1995].

Presentation of F_M when M is a matroid

Theorem [Baker-Lorscheid, 2023+]

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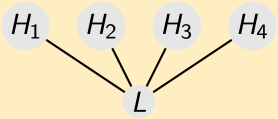

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Tutte's homotopy theorem

Let M be a matroid on E. Pick $a \in E$. Assume M and $M' = M \setminus a$ are connected.

The Tutte graph has vertex set $\{H \in \mathcal{H}' | a \notin \mathrm{cl}(H)\}$. Two vertices H_1 and H_2 are adjacent if H_1 and H_2 intersect at a connected corank 2 flat of M'.

Theorem [Tutte, 1958]

Every cycle in the Tutte graph can be decomposed into three different types of elementary cycles.

The first type $\longleftrightarrow$ Uniform$(2,4)$, Uniform$(3,5)$, or F_7.
The second type $\longleftrightarrow$ $M(K_4 - e)$.
The third type $\longleftrightarrow$ F_7^*.

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Thank you!